JOURNAL OF APPROXIMATION THEORY **58**, 232–241 (1989)

On Convergent Interpolatory Polynomials

P. ERDŐS, A. KROÓ,* AND J. SZABADOS*

Mathematical Institute, Hungarian Academy of Sciences, Reáltanoda u. 13-15, Budapest H-1053, Hungary

Communicated by Oved Shisha

Received December 2, 1987

Let

$$X_n\colon -1 \leqq x_{nn} < x_{n-1,n} < \cdots < x_{1n} \leqq 1 \qquad (n = 1, 2, ...) \tag{1}$$

be a system of nodes of interpolation. We are interested in finding necessary and sufficient conditions on (1) in order that for every $f(x) \in C[-1, 1]$ and $\varepsilon > 0$ there exist polynomials $p_n(x) \in \Pi_{[n(1+\varepsilon)]}$ such that

$$p_n(x_{kn}) = f(x_{kn}) \qquad (k = 1, ..., n;\ n = 1, 2, ...) \tag{2}$$

and

$$\lim_{n \to \infty} \|f(x) - p_n(x)\| = 0. \tag{3}$$

Here Π_m is the set of algebraic polynomials of degree at most m, $C[-1, 1]$ is the space of continuous functions on the interval $[-1, 1]$, and $\|\cdot\|$ is the maximum (over $[-1, 1]$) norm.

Let $x_{kn} = \cos t_{kn}$, $0 \leqq t_{1n} < t_{2n} < \cdots < t_{nn} \leqq \pi$, and for an arbitrary interval $I \subseteq [0, \pi]$, denote

$$N_n(I) = \sum_{t_{kn} \in I} 1.$$

In this paper we shall prove the following

THEOREM. *For every $f(x) \in C[-1, 1]$ and $\varepsilon > 0$ there exists a sequence of polynomials $p_n(x) \in \Pi_{[n(1+\varepsilon)]}$ such that* (2) *and*

$$\|f(x) - p_n(x)\| = O(E_{[n(1+\varepsilon)]}(f)) \tag{4}$$

* The research of these authors was supported by the Hungarian National Foundation for Scientific Research, Grant 1801.

0021-9045/89 $3.00

hold, if and only if

$$\overline{\lim_{n\to\infty}} \frac{N_n(I_n)}{n|I_n|} \leqq \frac{1}{\pi} \qquad \text{whenever} \quad \lim_{n\to\infty} n|I_n| = \infty \qquad (|I_n| = \text{length of } I_n) \tag{5}$$

and

$$\underline{\lim_{n\to\infty}} \min_{1\leqq i\leqq n-1} n(t_{i+1,n} - t_{i,n}) > 0. \tag{6}$$

Here the O sign refers to $n \to \infty$ and indicates a constant depending only on ε; $E_m(f)$ is the best uniform approximation of $f(x)$ by polynomials of degree at most n.

This theorem, in a slightly weaker form ((4) replaced by (3)) was stated in [1, Theorem 4]. There was no proof given, only an indication that it is a simple modification of the proof of Theorem 3. While we were unable to reconstruct this "simple modification" (it was probably not that simple at all), we found a proof which we think worthwhile to publish, since the above theorem is a fundamental and frequently quoted result of the theory of interpolation.

The proof is long and sophisticated, and in order to make it more understandable we break it into a series of lemmas. First we aim at the sufficiency of conditions (5)–(6).

LEMMA 1. *Under conditions* (5), (6) *for any* $\varepsilon > 0$ *there exists a system of nodes* (*in not necessarily decreasing order*)

$$\begin{aligned} Y_n : y_k = y_{kn} &= \cos \eta_k, \\ \eta_k = \eta_{km} &= \frac{2k - 1 + d_k}{m} \frac{\pi}{2}, \\ k = 1, \ldots, m &= [n(1+\varepsilon)];\ n \geqq n_0 \end{aligned} \tag{7}$$

such that

(a) *the* x_i's *are among the* y_k's;

(b) $n(\eta_{k+1} - \eta_k) \geqq c > 0$ $(k = 1, \ldots, m;\ n \geqq n_0)$ *with an absolute constant* c, *and*

(c) $|\sum_{k=1}^{s} d_k| \leqq A$ $(s = 1, \ldots, m)$ *with a constant* $A = A(\varepsilon)$.

Proof. Condition (5) implies that for any $\varepsilon > 0$, there exist $\Delta(\varepsilon)$ and $n_0(\varepsilon)$ such that

$$\frac{N_n(I)}{n|I|} \leqq \frac{1}{\pi} + \varepsilon \qquad \text{whenever} \quad n(I) \geqq \Delta(\varepsilon) \qquad \text{and} \qquad n \geqq n_0(\varepsilon). \tag{8}$$

Let

$$\Delta = \max\left(\Delta\left(\frac{\varepsilon}{4}\right), \frac{30}{\varepsilon}\right)$$

and consider the intervals

$$J_i = \left[\frac{i\Delta}{n}, \frac{(i+1)\Delta}{n}\right) \qquad \left(i = 0, ..., \left[\frac{\pi n}{\Delta}\right] - 1\right).$$

By (8) and $n|J_i| = \Delta$,

$$N_n(J_i) \leqq \left(\frac{1}{\pi} + \frac{\varepsilon}{4}\right)\Delta \qquad \left(i = 0, ..., \left[\frac{\pi n}{\Delta}\right] - 1\right).$$

The number of equidistant nodes

$$\theta_k = \frac{2k-1}{m+1}\frac{\pi}{2} \qquad (k = 1, ..., m+1)$$

in J_i is $\geqslant (\Delta(m+1))/\pi n > (\Delta/\pi)(1+\varepsilon)$, i.e., at least $\Delta\varepsilon(1/\pi - 1/4) > 3$ more than $N_n(J_i)$.

We shall construct the η_k's in two phases. In the first phase, in each J_i where at least one t_k occurs, replace the θ_j's by these t_k's, and leave the remaining θ_j's unchanged. According to the previous argument, there is at least one such unchanged θ_j in each J_i (call them free nodes). This system fulfils so far only (a). We would like to ensure (b). By (6) we may assume that

$$t_{i+1} - t_i \geqslant \frac{c}{n} \qquad (c < 1, \ i = 1, ..., n-1). \tag{9}$$

Consider those remaining θ_j's for which there exists a t_i such that

$$0 < |\theta_j - t_i| \leqslant \frac{c}{7n}, \tag{10}$$

and move these θ_j's alternatively to the left or to the right with a distance $2c/(7n)$. Then these translated θ_k's will be farther than $c/(7n)$ from any t_i

(see (9)), and the distance of adjacent new θ_j's will be at least $\pi/(m+1)-4c/(7n)>(\pi/2-4/7)(1/n)$. Thus the change in the contribution of the d_k's will be $O(1)$, and (b) is satisfied. After completing these steps, at least one free node remains in each J_i.

In the second phase we want to ensure (c) by further modifications. Divide consecutive J_i's into groups of 10Δ members. In each J_i, the maximal contribution of d_k's is $<(1/\pi+\varepsilon/4)\Delta\cdot 2(1+\varepsilon)\,\Delta/\pi<\Delta^2$ (we may assume that $\varepsilon<1$); thus for the whole group it is $<10\Delta^3$. We would like to arrive at a situation where the *total* contribution of d_k's at the end of each group is $<10\Delta^3$. We proceed by induction on the number of groups. As we have seen, in the first group the contribution is $<10\Delta^3$. Assume that the total contribution of the first $a-1$ groups is $<10\Delta^3$, and, without loss of generality we may assume that this contribution is nonnegative. By proper changes, we would like to have a contribution in the ath group between $-10\Delta^3$ and 0, thus ensuring a total contribution in the first a groups between $-10\Delta^3$ and $10\Delta^3$. In the ath group, the total contribution is between $-10\Delta^3$ and $10\Delta^3$. If it is negative, we are done. Thus assume that it is between 0 and $10\Delta^3$, and omit a free node from the $(5\Delta+2)$nd J_i and replace it by the midpoint of any two adjacent nodes in the $(5\Delta-2)$nd J_i. The result is a decrease of at least $2\cdot 2(1+\varepsilon)\,\Delta/\pi$ and at most $4\cdot 2(1+\varepsilon)\,\Delta/\pi$ in the contribution of the d_k's in the ath group. If this change transforms this contribution below zero, then we are done. If not, then omit a free node from the $(5\Delta+3)$rd J_i and replace it by the midpoint of any two adjacent nodes in the $(5\Delta-3)$rd J_i. The result is another decrease of at least $4\cdot 2(1+\varepsilon)\,\Delta/\pi$ and at most $6\cdot 2(1+\varepsilon)\,\Delta/\pi$ in the contribution of the d_k's in the ath group. If this second change transforms this contribution below zero, then we are done; otherwise continue this procedure with the $(5\Delta+4)$th and $(5\Delta-4)$th J_i's, etc. Before exhausting all the possibilities we must arrive at the desired situation, because the decrease of the contribution in the ath group after all the possible changes would be at least

$$(2+4+\cdots+10\Delta-2)(1+\varepsilon)\,\Delta/\pi>\frac{2\Delta}{\pi}\,5\Delta(5\Delta-1)>\frac{40\Delta^3}{\pi}$$

which is greater than $10\Delta^3$, the original maximal contribution in the ath group. (Even if we needed the last change here, its maximal contribution is $<10\Delta\cdot 2(1+\varepsilon)\,\Delta/\pi<13\Delta^2<10\Delta^3$, so we never get under $-10\Delta^3$.)

After making all these changes in each group, we arrive at a situation where the total contribution of the d_k's *at the last J_i in a group* will be $<10\Delta^3$. But it is clear from the previous argument that $|d_k|<13\Delta^2$, and since the number of d_k's in a group is $<10\Delta\cdot(\Delta(1+\varepsilon)/\pi)+5\Delta<12\Delta^2$, the contribution *inside* a group cannot be higher than $13\Delta^2 \cdot 12\Delta^2$, i.e., bounded again. Thus Lemma 1 is completely proved. ∎

LEMMA 2. *For the fundamental functions of Lagrange interpolation based on the nodes* (7) *we have*

$$\|l_j(Y_m, x)\| = O(1) \qquad (k = 1, ..., m).$$

Proof. Let

$$Z_m : z_k = \cos\frac{2k-1}{2m}\pi \qquad (k = 1, ..., m);$$

$$T_m(x) = \prod_{k=1}^{m} (x - z_k), \tag{11}$$

$$\Omega_m(x) = \prod_{k=1}^{m} (x - y_k).$$

Then for a fixed k, the number ν_k of y_i's for which $\operatorname{sgn}(y_k - y_i) = \operatorname{sgn}(k - i)$ is evidently $\nu_k = o(1)$, and thus denoting $A_k = \{i \mid \operatorname{sgn}(y_k - y_i) = \operatorname{sgn}(k - i)\}$, $B_k = \{1, ..., m\} \backslash A_k$ we have

$$\begin{aligned}\left|\frac{T'_m(z_k)}{\Omega'(y_k)}\right| &= \prod_{i \in A_k} \frac{z_k - z_i}{y_i - y_k} \prod_{i \in B_k} \frac{z_k - z_i}{y_k - y_i} \\ &= O(1) \prod_{i \in B_k} \left(1 + \frac{z_k - y_k + y_i - z_i}{y_k - y_i}\right) \\ &= O(1) \exp \sum_{i \in B_k} \frac{z_k - y_k + y_i - z_i}{y_k - y_i} \\ &= O(1) \exp \sum_{i \neq k} \frac{z_k - y_k + y_i - z_i}{y_k - y_i}.\end{aligned}$$

Here, using $|d_k| = O(1)$ (see Lemma 1(c)), we get for $1 \leqq k \leqq m/2$

$$\begin{aligned}&|z_k - y_k| \sum_{i \neq k} \frac{1}{y_k - y_i} \\ &= O\left(\frac{k\,|d_k|}{m^2}\right)\left\{\left|\sum_{i \neq k} \frac{1}{z_i - z_k}\right| + \sum_{i \neq k} \left|\frac{y_k - z_k + z_i - y_i}{(z_k - z_i)(y_k - y_i)}\right|\right\} \\ &= O\left(\frac{k}{m^2}\right)\left\{\left|\frac{T''_m(z_k)}{T'_m(z_k)}\right| + \sum_{i \neq k} \frac{(k|d_k|/m^2) + (i|d_i|/m^2)}{((k-i)^2 \min(k+i, m/2)^2)/m^4}\right\} \\ &= O\left(\frac{k}{m^2}\right)\left\{\frac{m^2}{k^2} + \frac{m^2}{k} \sum_{i \neq k} \frac{1}{(k-i)^2}\right\} = O(1),\end{aligned}$$

and using Abel's transform

$$\left|\sum_{i\neq k}\frac{z_i-y_i}{y_k-y_i}\right|=\left|\sum_{i\neq k}\frac{2\sin(d_i\pi/4m)\sin((4i-2+d_i)/4m)\pi}{y_k-y_i}\right|$$

$$=\left|\sum_{i\neq k}\frac{(d_i\pi/2m)\sin((4i-2+d_i)/4m)\pi+O(m^{-3})}{y_k-y_i}\right|$$

$$=O\left\{\frac{1}{m}\sum_{i\neq k,k+1}\left(\frac{\sin((4i+2+d_i)/4m)\pi}{y_k-y_{i+1}}-\frac{\sin((4i-2+d_i)/4m)\pi}{y_k-y_i}\right)\sum_{j=1}^{i}d_j\right\}+O(1)$$

$$=O\left(\frac{1}{m}\right)\cdot\sum_{i\neq k,k+1}\left(\frac{(i/m)|y_i-y_{i+1}|}{|y_k-y_{i+1}|\cdot|y_k-y_i|}+\frac{i/m^2}{|y_k-y_i|}\right)+O(1)$$

$$=O\left(\frac{1}{m}\right)\sum_{i\neq k}\left(\frac{i^2/m^3}{(k^2-i^2)^2/m^4}+\frac{i/m^2}{|k^2-i^2|/m^2}\right)+O(1)$$

$$=O\left(\sum_{i\neq k}\frac{1}{(k-i)^2}+\frac{1}{m}\sum_{i\neq k}\frac{1}{|k-i|}+1\right)=O(1),$$

and similarly for $m/2\leqq k\leqq m$. Hence

$$|T'_m(z_k)|=O(|\Omega'_m(y_k)|)\qquad(k=1,\ldots,m).\tag{12}$$

Now let $|x|\leqq 1$ be arbitrary and $0\leqq j\leqq m$ be such that $z_{j+1}\leqq x\leqq z_j$ (we take $z_0=1$ and $z_{m+1}=-1$). Then similarly as before, denoting $u\in(z_{j+1},z_j)$ for which $T_m(u)$ is a local maximum, the number $v(x)$ of i's for which $\operatorname{sgn}((x-y_i)/(u-z_i))=-1$ is evidently $v(x)=O(1)$. Hence

$$\left|\prod_{i=k}\frac{x-y_i}{u-z_i}\right|=\prod_{\operatorname{sgn}((x-y_i)/(u-z_i))=-1}\left|\frac{x-y_i}{u-z_i}\right|$$

$$\times\prod_{\operatorname{sgn}((x-y_i)/(u-z_i))\geqq 0}\left(1+\frac{x-u+z_i-y_i}{u-z_i}\right)$$

$$=O(1)\exp\sum_{\operatorname{sgn}((x-y_i)/(u-z_i))\geqq 0}\frac{x-u+z_i-y_i}{u-z_i}$$

$$=O(1)\exp\left\{|x-u|\left(\left|\frac{T'_m(u)}{T_m(u)}\right|+\sum_{\operatorname{sgn}((x-y_i)/(u-z_i))=-1}\frac{1}{|u-z_i|}\right)+\sum_{i\neq j}\frac{i/m^2}{|j^2-i^2|/m^2}\right\}$$

$$=O(1)\exp O\left\{\frac{j}{m^2}\left(\frac{m^2}{j}+v(x)\cdot\frac{m^2}{j}\right)+\frac{1}{j}\right\}=O(1).$$

Thus using (12) we get

$$\left|\frac{l_k(Y_m, x)}{l_k(Z_m, u)}\right| = \left|\frac{T'_m(z_k)}{\Omega'_m(y_k)} \prod_{i \neq k} \frac{x - y_i}{u - z_i}\right| = O(1) \qquad (k = 1, ..., m);$$

i.e., using Fejér's result $\|l_k(z_m, u)\| \leqq \sqrt{2}$ $(k = 1, ..., m)$ we get the statement of the lemma. ∎

After these preliminaries, the sufficiency of conditions (5), (6) is easily proved. Let $s = [n\varepsilon/3]$, and apply Lemma 1 with $\varepsilon/3$ instead of ε; then $m = [n(1 + \varepsilon/3)]$. Let $g(x) \in \Pi_{[n(1+\varepsilon)]}$ be the best approximating polynomial of $f(x)$. Consider

$$p_n(x) = q(x) + \sum_{j=0}^{s} \left\{ \sum_{z_{j+1,s} < y_k \leqq z_{j,s}} \frac{(f(x_k) - q(x_k))\, l_k(Y_m, x)}{\{l_j(Z_s, y_k) + l_{j+1}(Z_s, y_k)\}^2} \right.$$
$$\left. \times \{l_j(Z_s, x) + l_{j+1}(Z_s, x)\}^2 \right\}.$$

Since by the well-known Erdős–Turán result [2, Lemma IV]

$$l_j(Z_s, y_k) + l_{j+1}(Z_s, y_k) \geqq 1 \qquad (z_{j+1} < y_k \leqq z_j), \tag{13}$$

the definition of $p_n(x)$ makes sense. Now

$$\deg p_n \leqq m - 1 + 2(s - 1) < n\left(1 + \frac{\varepsilon}{3}\right) + \frac{2n\varepsilon}{3} = n(1 + \varepsilon),$$

and evidently

$$p_n(y_i) = f(y_i) \qquad (i = 1, ..., m).$$

This proves (2), since by Lemma 1(a) the x_k's are among the y_i's. By the definition of $q(x)$, (13), Lemma 2, and the inequality $(a+b)^2 \leqq 2(a^2 + b^2)$ we get

$$\|f(x) - p_n(x)\| \leqq \|f(x) - q(x)\| \left\{1 + O\left[\left\|\sum_{j=0}^{s} l_j(Z_s, x)^2 \sum_{z_{j+1} < y_k \leqq z_j} 1\right\|\right]\right\}$$
$$= O(E_{[n(1+\varepsilon)]}(f)) \left\|\sum_{j=0}^{s} l_j(Z_s, x)^2\right\|,$$

since by Lemma 1(b), $\sum_{z_{j+1} < y_k \leqq z_j} 1 = O(1)$. But here again by Fejér's result

$$\left\|\sum_{j=0}^{s} l_j(Z_s, x)^2\right\| \leqq 2$$

and thus (4) is also proved.

To prove the necessity of (6), assume that there exists a sequence $i_1 < i_2 < \cdots$ such that

$$\lim_{n \to \infty} n(t_{i_n+1,n} - t_{i_n,n}) = 0.$$

Hence passing to monotone subsequences (if necessary), there exists a $t \in [0, \pi]$ such that

$$\lim_{n \to \infty} t_{i_n,n} = t, \qquad t_{i_n+1,n} - t_{i_n,n} \leqq \frac{\varepsilon_n}{n}, \qquad \lim_{n \to \infty} \varepsilon_n = 0, \tag{14}$$

and the sequences $\{t_{i_n,n}\}$ and $\{t_{i_n+1,n}\}$ have no points in common. Also, we may assume that at least one of these sequences, say $\{t_{i_n,n}\}$, is strictly monotone. Then define

$$f(t_{i_n,n}) = 0, \qquad f(t_{i_n+1,n}) = \sqrt{\varepsilon_n},$$

and f is continuous and linear between these nodes. Because of (14), this defines an $f(x) \in C[-1, 1]$. By (2) and the Bernstein inequality

$$\begin{aligned}
\frac{n}{\sqrt{\varepsilon_n}} &\leqq \frac{f(\cos t_{i_n+1,n}) - f(\cos t_{i_n,n})}{t_{i_n+1,n} - t_{i_n,n}} \\
&= \frac{p_n(\cos t_{i_n+1,n}) - p_n(\cos t_{i_n,n})}{t_{i_n+1,n} - t_{i_n,n}} \\
&= \frac{d}{dt} p_n(\cos t) \bigg|_{t=\xi} = O(n) \|p_n\| \qquad (\xi \in (t_{i_n,n}, t_{i_n+1,n})),
\end{aligned}$$

i.e., $\|p_n\| \geqq 1/\sqrt{\varepsilon_n} \to \infty$ as $n \to \infty$, which shows that (4) cannot hold. Hence (6) is necessary.

The proof of the necessity of (5) is more difficult. First we prove the following.

LEMMA 3. *Let $I_n \subset [-\pi, \pi]$ $(n \in \mathbb{N})$ and let t_n be a sequence of trigonometric polynomials of order at most r_n such that $r_n |I_n| \to \infty$ and $\|t_n\| \leqslant M$ $(n \in \mathbb{N})$ $(r_n \uparrow \infty)$. Denote by $Q(I_n)$ the number of $+1, -1, +1, \ldots$ oscillations of t_n on I_n. Then*

$$\overline{\lim_{n \to \infty}} \frac{Q(I_n)}{r_n |i_n|} \leqslant \frac{1}{\pi}.$$

Proof. Assume to the contrary that $Q(I_n)/r_n |I_n| > (1 + \delta)/\pi$ for some $\delta > 0$ and $n \in \Omega$ $(\Omega \subset \mathbb{N}$ infinite), where we may take $I_n(-a_n, a_n)$ and $0 < a_n < \pi - 2\delta_1$. Let now s_n be an even integer such that

$\sqrt{r_n a_n} < s_n < 2\sqrt{r_n a_n}$ and let $\varepsilon_n = \pi M a_n/(s_n \sin \delta_1)$. Consider the trigonometric polynomial

$$u_n(x) = t_n(x) + \frac{1}{2}\left(\frac{\sin(x/2)}{\sin(a_n/2)}\right)^{s_n} \cos\left(r_n - \frac{s_n}{2}\right) x.$$

of order at most r_n. Evidently, on $[-a_n, a_n]$, u_n has at least $Q(I_n) - 1$ zeros. If $x \notin (-a_n - \varepsilon_n, a_n + \varepsilon_n)$ we have for s_n large enough

$$\begin{aligned}\left(\frac{\sin(x/2)}{\sin(a_n/2)}\right)^{s_n} &\geqslant \left(\frac{\sin((a_n + \varepsilon_n)/2)}{\sin(a_n/2)}\right)^{s_n} \\ &= \left(1 + \frac{2\sin(\varepsilon_n/4)\cos(a_n/2 + \varepsilon_n/4)}{\sin(a_n/2)}\right)^{s_n} \\ &\geqslant \left(1 + \frac{2\varepsilon_n \sin \delta_1}{\pi a_n}\right)^{s_n} \\ &= \left(1 + \frac{2M}{s_n}\right)^{s_n} \geqslant 2^{2M} > 2M.\end{aligned}$$

Thus u_n has at least $(2\pi - 2a_n - 2\varepsilon_n)((2r_n - s_n)/2\pi) - 4$ zeros in $[-\pi, \pi] \backslash (-a_n - \varepsilon_n, a_n + \varepsilon_n)$. Therefore

$$Q(I_n) + (2\pi - 2a_n - 2\varepsilon_n)\frac{2r_n - s_n}{2\pi} \leqslant 2r_n + 5,$$

i.e.,

$$Q(I_n) \leqslant 5 + \frac{2a_n r_n}{\pi} + \frac{2\varepsilon_n r_n}{\pi} + s_n,$$

$$\frac{1+\delta}{\pi} < \frac{Q(I_n)}{r_n |I_n|} = \frac{Q(I_n)}{2r_n a_n} \leqslant \frac{1}{\pi} + c\left(\frac{1}{r_n a_n} + \frac{\varepsilon_n}{a_n} + \frac{1}{\sqrt{r_n a_n}}\right),$$

a contradiction, since $r_n a_n \to \infty$ and $\varepsilon_n/a_n = c/s_n \to 0$.

We now return to the proof of the necessity of (5). Define the continuous 2π-periodic function F_n by $F_n(t_{kn}) = (-1)^k$ $(1 \leqslant k \leqslant n)$, F_n is linear in between, constant in $[0, t_{1n}]$, $[t_{nn}, \pi]$, $F_n(t) = F_n(-t)$ $(-\pi \leqslant t \leqslant 0)$, and $F_n(t + 2\pi) = F_n(t)$ $(-\infty < t < \infty)$. By (15) $\omega(F_n, h) \leqslant cnh$, hence $E_n^T(F_n) \leqslant c_1$. Set $f_n(x) = F_n(\arccos x)$. Then by assumption for any $\varepsilon > 0$ there exist $p_n \in \Pi_{[(1+\varepsilon)n]}$ such that $p_n(x_{kn}) = f_n(x_{kn}) = (-1)^k$ $(1 \leqslant k \leqslant n)$ and

$$\|f_n - p_n\| \leqslant c_\varepsilon E_{[(1+\varepsilon)n]}(f_n) = c_\varepsilon E^T_{[(1+\varepsilon)n]}(F_n) \leqslant \tilde{c}_\varepsilon.$$

Thus $\|p_n\| \leqslant c_\varepsilon^*$ (deg $p_n = [(1+\varepsilon)n]$); hence by Lemma 3

$$\overline{\lim_{n\to\infty}} \frac{N_n(I_n)}{[(1+\varepsilon)n]\,|I_n|} \leqslant \overline{\lim_{n\to\infty}} \frac{Q(I_n)}{[(1+\varepsilon)n]\,|I_n|} \leqslant \frac{1}{\pi}.$$

Since $\varepsilon > 0$ is arbitrary, we can put $\varepsilon = 0$ here.

Using the same arguments, we could have proved the following, slightly more general theorem:

THEOREM A. *For every $f(x) \in C[-1, 1]$, $\varepsilon > 0$, and $d \geqslant 1$ there exists a sequence of polynomials $q_n(x) \in \Pi_{[dn(1+\varepsilon)]}$ such that* (2) *and*

$$\|f(x) - q_n(x)\| = O(E_{[dn(1+\varepsilon)]}(f))$$

hold, if and only if

$$\overline{\lim_{n\to\infty}} \frac{N_n(I_n)}{n|I_n|} \leqq \frac{d}{\pi} \qquad \textit{whenever} \quad \lim_{n\to\infty} n|I_n| = \infty$$

and (6) *holds.*

REFERENCES

1. P. ERDŐS, On some convergence properties of the interpolation polynomials, *Ann. of Math.* (2) **44** (1943), 330–337.
2. P. ERDŐS AND P. TURÁN. On interpolation III, *Ann. of Math.* (2) **41** (1940), 510–553.
3. M. RIESZ, Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome. *Jahresber. Deutsch. Math.-Verein.* **23** (1914), 354–368.